Optimization – Exercises

Day 1

Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space. We denote $\| \cdot \|$ the norm derived by the scalar product.

Exercise 1 (Necessary and sufficient optimality conditions).
Let $f\colon H \longrightarrow \mathbb{R}$ be a twice differentiable function. Show that if x is a local minimizer of f, then

$$\nabla f(x) = 0$$
$$\nabla^2 f(x) \geq 0$$

Is the first order condition a sufficient condition for x to be a local minimizer ? If no, give an example. What assumption can you make for this condition to be an equivalence ?

Correction.

- ▷ First order necessary condition. Suppose x is a local minimizer of f. Then there exists $r > 0$ such that for all $y \in \mathcal{B}(x, r)$, $f(y) \geqslant f(x)$. Let $h \in H$ and $t > 0$ such that $x + th \in \mathcal{B}(x, r)$. We have

$$f(x + th) = f(x) + \langle \nabla f(x), th \rangle + o(t).$$

Then

$$\langle \nabla f(x), h \rangle + o(1) = \frac{f(x + th) - f(x)}{t}.$$

Since $f(x + th) - f(x) \geqslant 0$ and $t > 0$, we get

$$\langle \nabla f(x), h \rangle + o(1) \geqslant 0.$$

Letting t tend toward 0^+,

$$\langle \nabla f(x), h \rangle \geqslant 0.$$

The same reasoning can be done with $t < 0$, yielding

$$\langle \nabla f(x), h \rangle \leqslant 0.$$

Finally, forall $h \in H$,

$$\langle \nabla f(x), h \rangle = 0,$$

thus $\nabla f(x) = 0$.

- ▷ Second order necessary condition. Suppose x is a local minimizer of f. Then there exists $r > 0$ such that for all $y \in \mathcal{B}(x, r)$, $f(y) \geqslant f(x)$. Let $h \in H$ and $t > 0$ such that $x + th \in \mathcal{B}(x, r)$.
Using second order Taylor-Young's expansion :

$$\begin{aligned} f(x + th) &= f(x) + t\langle \nabla f(x), h \rangle + \frac{t^2}{2}\langle h, \nabla^2 f(x) h \rangle + o(t^2) \\ &= f(x) + \frac{t^2}{2}\langle h, \nabla^2 f(x) h \rangle + o(t^2), \end{aligned}$$

where we used the first order necessary condition. Dividing the inequality by $t^2/2$ and letting t tend toward 0, it follows form the fact that x is a local mimimizer that

$$\langle h, \nabla^2 f(x)h\rangle \geqslant 0.$$

This last inequality is true for all $h \in H$ and proves the property.

The converse property does not hold generally : consider $f : x \mapsto x^3$ and $x = 0$ for instance. **The first order condition becomes a necessary and sufficient condition when f is convexe. Moreover, il this case, local minimizers are global mimimizers.**

Exercise 2 (Caracterizations of convex functions).

Let $f : H \longrightarrow \mathbb{R}$ be a twice differentiable function. Show the following equivalences :

1. f is convex if, and only if,

$$\forall (x, y) \in H \times H,\ f(y) \geqslant f(x) + \langle \nabla f(x) \mid y - x\rangle.$$

2. f is convex if, and only if,

$$\forall x \in H,\ \nabla^2 f(x) \geq 0,$$

where $\nabla^2 f(x)$ is the hessian of f at x.

Correction.

1. $\boxed{\Rightarrow}$ Suppose f is convex. Let $(x, y) \in H \times H$. Since f is differentiable, we have for all $t \in]0, 1]$:

$$f(x + t(y - x)) = f(x) + t\,\langle \nabla f(x), y - x\rangle + o(t).$$

Moreover, using the convexity of f,

$$f(x + t(y - x)) \leqslant (1 - t)f(x) + tf(y).$$

It follows that

$$tf(y) \geqslant tf(x) + t\,\langle \nabla f(x), y - x\rangle + o(t).$$

Dividing the inequality by $t > 0$ and letting t tend toward 0, we finally get

$$f(y) \geqslant f(x) + \langle \nabla f(x), y - x\rangle.$$

$\boxed{\Leftarrow}$ Let $(x, y) \in H \times H$ and $t \in [0, 1]$. Let $z = x + t(y - x)$. We have

$$f(x) - f(z) \geqslant \langle \nabla f(z), -t(y - x)\rangle$$
$$f(y) - f(z) \geqslant \langle \nabla f(z), (1 - t)(y - x)\rangle.$$

Multiplying the first inequality by $(1 - t)$ and the second by t, we get

$$(1 - t)f(x) + tf(y) - f(z) \geqslant 0,$$

which is the desired result.

2. $\boxed{\Rightarrow}$ Suppose f is convex. Let $x \in H$, $h \in H$, $t > 0$. It follows form second order Taylor-Young's formula :

$$f(x + th) - f(x) - t\langle \nabla f(x), h\rangle = \frac{t^2}{2}\langle h, \nabla^2 f(x)h\rangle + o(t^2) \geqslant 0.$$

Simplifying by $t^2/2$, and letting t tend toward 0, we finally get

$$\langle h, \nabla^2 f(x) h\rangle \geqslant 0.$$

$\boxed{\Leftarrow}$ ⚠ We did not suppose f is a twice continuously differentiable function. Thus, we cannot use Taylor in its integral from. However, we can apply second order Taylor-Lagrange's expansion to the function $\Phi : t \mapsto f(x + t(y - x))$: there exists $t^* \in [0, 1]$ such that

$$\Phi(1) = \Phi(0) + \Phi'(0) + \frac{1}{2}\Phi''(t^*),$$

i.e.

$$f(y) = f(x) + \langle \nabla f(x), y - x\rangle + \frac{1}{2}\langle y - x, \nabla^2 f(x + t^*(y - x))(y - x)\rangle.$$

The hypothesis $\nabla^2 f \geq 0$ finally gives

$$f(y) - f(x) \geqslant \langle \nabla f(x), y - x\rangle,$$

which is equivalent to the convexity of f.

Exercise 3 (Squared distance function).
Let A be a nonempty closed convex subset of H. We consider the function "squared distance to A" defined for all $x \in H$ by

$$g(x) = \inf_{y \in A} \|x - y\|^2.$$

1. Show that g is convex.
2. Show that g is Fréchet differentiable, with $\nabla g(x) = 2(x - \mathrm{p}_A(x))$, where p_A denotes the projection on A.

Correction.

1. Let $(x_1, x_2) \in H^2$ and $t \in [0, 1]$. Since A is nonempty closed and convex, the projection p_A is well defined. Using that $t\mathrm{p}_A(x_1) + (1 - t)\mathrm{p}_A(x_2) \in A$, it follows that

$$\begin{aligned} g(tx_1 + (1 - t)x_2) &\leqslant \|tx_1 + (1 - t)x_2 - (t\mathrm{p}_A(x_1) + (1 - t)\mathrm{p}_A(x_2))\|^2, \\ &= \|t(x_1 - \mathrm{p}_A(x_1)) + (1 - t)(x_2 - \mathrm{p}_A(x_2))\|^2, \\ &\leqslant t\|x_1 - \mathrm{p}_A(x_1)\|^2 + (1 - t)\|x_2 - \mathrm{p}_A(x_2)\|^2, \\ &= tg(x_1) + (1 - t)g(x_2). \end{aligned}$$

2. Let $(x, h) \in H$,

$$\begin{aligned} g(x + h) &= \|(x + h) - \mathrm{p}_A(x + h)\|^2, \\ &= \|x - \mathrm{p}_A(x) + \mathrm{p}_A(x) - \mathrm{p}_A(x + h) + h\|^2, \\ &= g(x) + 2\langle x - \mathrm{p}_A(x), \mathrm{p}_A(x) - \mathrm{p}_A(x + h) + h\rangle + \|\mathrm{p}_A(x) - \mathrm{p}_A(x + h) + h\|^2, \\ &= g(x) + \langle 2(x - \mathrm{p}_A(x)), h\rangle + \theta(x, h), \end{aligned}$$

where

$$\theta(x, h) = 2\langle x - \mathrm{p}_A(x), \mathrm{p}_A(x) - \mathrm{p}_A(x + h)\rangle + \|\mathrm{p}_A(x) - \mathrm{p}_A(x + h) + h\|^2.$$

Let us prove that $\theta(x,h) = o(\|h\|)$. By definition of the gradient operator, this will conclude the proof. First, recall the following characterization of the projection :
Property. For all $x \in H$,

$$\boxed{\forall y \in A, \quad \langle x - \mathrm{p}_A(x), y - \mathrm{p}_A(x)\rangle \leqslant 0}$$

Using this property, we deduce that

$$0 \leqslant \theta(x,h).$$

Moreover,

$$\begin{aligned}
\theta(x,h) &= 2\langle x - \mathrm{p}_A(x+h) + \mathrm{p}_A(x+h) - \mathrm{p}_A(x), \mathrm{p}_A(x) - \mathrm{p}_A(x+h)\rangle \\
&\quad + \|\mathrm{p}_A(x) - \mathrm{p}_A(x+h) + h\|^2 \\
&\leqslant 2\langle x - \mathrm{p}_A(x+h), \mathrm{p}_A(x) - \mathrm{p}_A(x+h)\rangle + \|\mathrm{p}_A(x) - \mathrm{p}_A(x+h) + h\|^2, \\
&= 2\langle x + h - \mathrm{p}_A(x+h), \mathrm{p}_A(x) - \mathrm{p}_A(x+h)\rangle - 2\langle h, \mathrm{p}_A(x) - \mathrm{p}_A(x+h)\rangle \\
&\quad + \|\mathrm{p}_A(x) - \mathrm{p}_A(x+h) + h\|^2 \\
&\leqslant -2\langle h, \mathrm{p}_A(x) - \mathrm{p}_A(x+h)\rangle + \|\mathrm{p}_A(x) - \mathrm{p}_A(x+h) + h\|^2.
\end{aligned}$$

Finally, using Cauchy-Schwarz inequality and the following well known property of the projection, one easily derives that $0 \leqslant \theta(x,h) \leqslant \mathrm{cst}\,\|h\|^2$.
Property. For all $(x,y) \in H^2$,

$$\|\mathrm{p}_A(x) - \mathrm{p}_A(y)\| \leqslant \|x - y\|.$$

Exercise 4 (Minimization of a quadratic function).
Let $A \in \mathcal{S}_n^{++}(\mathbb{R})$ (set of symmetric positive definite matrices of $\mathbb{R}^{n\times n}$) and $b \in \mathbb{R}^n$. Let f be defined for all $x \in \mathbb{R}^n$ by

$$f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle.$$

Show that f admits a unique minimizer and give an expression of this minimizer.

Correction.

- **Existence**. f is clearly lower-semi-continuous and proper. We show that f is coercive. Since $A \in \mathcal{S}_N^{++}(\mathbb{R})$, we have

$$f(x) \geqslant \frac{1}{2}\lambda_{\min}\|x\|^2 - \langle b, x\rangle,$$

  where $\lambda_{\min}$ is the smallest eigenvalue of A. It follows from Cauchy-Schwarz inequality

$$f(x) \geqslant \frac{1}{2}\lambda_{\min}\|x\|^2 - \|b\|\|x\| \underset{\|x\|\to+\infty}{\longrightarrow} +\infty.$$

  We conclude that f admits at least one global minimizer.
- **Unicity**. f is strictly convex on $\mathbb{R}^n$ since for all $x \in \mathbb{R}^n$,

$$\nabla^2 f(x) = A \succ 0.$$

  The minimizer is thus unique.
- **Expression of the minimizer**. Since f is a convex function, the first order condition is necessary and sufficient : x^* is a minimizer of f if, and only if, $\nabla f(x^*) = 0$. It follows that the unique minimizer x^* of f is given by

$$x^* = A^{-1}b.$$

Exercise 5 (Convex optimization exam 2019).
Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex, differentiable and bounded function on $\mathbb{R}^n$. Show f is constant.

Correction. We shall show that for all $x \in \mathbb{R}^n$, $\nabla f(x) = 0$. It is sufficient to show that for all $h \in \mathbb{R}^n$,

$$\langle \nabla f(x), h \rangle \leqslant 0.$$

Let $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$. For all $t > 0$, it follows from the convexity of f on $\mathbb{R}^n$ that :

$$f(x + th) - f(x) \geqslant t\langle \nabla f(x), h \rangle,$$

then

$$\frac{f(x + th) - f(x)}{t} \geqslant \langle \nabla f(x), h \rangle.$$

Letting t tend toward $+\infty$ and using the fact that f is bounded, we finally get

$$0 \geqslant \langle \nabla f(x), h \rangle.$$

Exercise 6 (About ε-minimizers).
Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function bounded from below on $\mathbb{R}^n$. Let $\varepsilon > 0$ and u a ε-minimizer of f, i.e. u satisfies

$$f(u) \leqslant \inf_{x \in \mathbb{R}^n} f(x) + \varepsilon.$$

Let $\lambda > 0$ and consider

$$g\colon x \in \mathbb{R}^n \mapsto g(x) := f(x) + \frac{\varepsilon}{\lambda}\|x - u\|.$$

1. Show there exists $v \in \mathbb{R}^n$ which minimizes g on $\mathbb{R}^n$. Show this point v satisfies the following conditions :
   (i) $f(v) \leqslant f(u)$,
   (ii) $\|u - v\| \leqslant \lambda$,
   (iii) $\forall x \in \mathbb{R}^n$, $f(v) \leqslant f(x) + \frac{\varepsilon}{\lambda}\|x - v\|$.
2. Suppose in addition that f is differentiable on $\mathbb{R}^n$. Show that for all $\epsilon > 0$, there exists $x_\epsilon \in \mathbb{R}^n$ such that

$$\|\nabla f(x_\epsilon)\| \leqslant \epsilon.$$

Correction.

1. Function g is continuous, and it is clear that $\lim_{\|x\| \to +\infty} g(x) = +\infty$ since f is bounded from below. Hence g admits a minimizer $v \in \mathbb{R}^n$.
   (i) By definition of v, for all $x \in \mathbb{R}^n$,

$$f(v) + \frac{\varepsilon}{\lambda}\|v - u\| \leqslant f(x) + \frac{\varepsilon}{\lambda}\|x - u\|. \tag{1}$$

   In particular, for $x = u$, we obtain $f(x) + \frac{\varepsilon}{\lambda}\|v - u\| \leqslant f(u)$. Therefore $f(v) \leqslant f(u)$.
   (ii) Denote by $\overline{f} = \inf_{x \in \mathbb{R}^n} f(x)$. Then according to (1),

$$\overline{f} + \frac{\varepsilon}{\lambda}\|v - u\| \leqslant f(u) \leqslant \overline{f} + \varepsilon,$$

   which directly implies that $\|v - u\| \leqslant \lambda$.
   (iii) From the reverse triangular inequality, $\|x - u\| - \|v - u\| \leqslant \|x - v\|$. Now, using (1), it follows that for all $x \in \mathbb{R}^n$,

$$f(v) \leqslant f(x) + \frac{\varepsilon}{\lambda}\|x - v\|.$$

2. Let $\epsilon > 0$. Fix $\lambda = \epsilon$ and $\varepsilon = \epsilon^2$. According to the previous question, there exists $x_\epsilon \in \mathbb{R}^n$ such that

$$\forall x \in \mathbb{R}^n, \; f(x_\epsilon) \leqslant f(x) + \epsilon \|x - x_\epsilon\|.$$

For $d \in \mathbb{R}^n$ and $\alpha > 0$, applying the previous inequality to $x = x_\epsilon + \alpha d$ and $x = x_\epsilon - \alpha d$ yields

$$\frac{f(x_\epsilon + \alpha d) - f(x_\epsilon)}{\alpha} \geqslant -\epsilon \|d\|,$$

and

$$\frac{f(x_\epsilon - \alpha d) - f(x_\epsilon)}{\alpha} \geqslant -\epsilon \|d\|.$$

Letting $\alpha \to 0^+$, it follows that

$$\langle \nabla f(x_\epsilon), d \rangle \geqslant -\epsilon \|d\| \text{ and } \langle \nabla f(x_\epsilon), -d \rangle \geqslant -\epsilon \|d\|,$$

i.e.

$$|\langle \nabla f(x_\epsilon), d \rangle| \leqslant \epsilon \|d\|.$$

This implies that $\|\nabla f(x_\epsilon)\| \leqslant \epsilon$.

Exercise 7.

Let $\mathcal{O} = \mathcal{S}_n^{++}(\mathbb{R})$ be the (open) set of symmetric positive definite matrices of $\mathbb{R}^{n \times n}$. $\mathcal{O}$ is endowed with the scalar product $\langle U, V \rangle = \text{Tr}(UV)$. Let $A \in \mathcal{O}$ and f be defined for all $X \in \mathcal{O}$ by

$$f(X) = \text{Tr}(X^{-1}) + \text{Tr}(AX).$$

1. Show there exists a minimizer to f on $\mathcal{O}$. *Hint : you may use the inequality* $\text{Tr}(UV) \geqslant \sum_{i=1}^n \lambda_i(U) \lambda_{n-i+1}(V)$, *where all eigenvalues* $\lambda_1, \ldots, \lambda_n$ *are in descending order ; i.e.,* $\lambda_1 \geqslant \cdots \geqslant \lambda_n$.
2. Find the minimizer and the optimal value of f.

Correction.

1. ▷ **Continuity**. f is continuous as a composition of continuous functions.
   ▷ **Coercivity**. We need to show that (a) $\lim\limits_{\substack{\|X\| \to +\infty \\ X \in \mathcal{O}}} f(X) = +\infty$ and (b) for all $\overline{X} \in \partial \mathcal{O}$, $\lim\limits_{\substack{X \to \overline{X} \\ X \in \mathcal{O}}} f(X) = +\infty$.
   (a) is clear since $f(X) \geqslant \text{Tr}(AX) \geqslant \sum_{i=1}^n \lambda_i(A) \lambda_{n-i+1}(X) \underset{\|X\| \to +\infty}{\longrightarrow} +\infty$.
   (b) Let $\overline{X} \in \partial \mathcal{O}$. Then $\lambda_n(\overline{X}) = 0$. If $\|X - \overline{X}\| \longrightarrow 0$, then $\lambda_n(X) \longrightarrow 0^+$. Hence,

$$f(X) \geqslant \text{Tr}(X^{-1}) = \sum_{i=1}^n \frac{1}{\lambda_i(X)} \underset{\|X - \overline{X}\| \to 0}{\longrightarrow} +\infty. \tag{2}$$

   Therefore, f admits a global minimizer.
2. If X^* is a minimizer of f on $\mathcal{O}$, then $\mathrm{d}f(X^*) = 0$. We first start computing the differential of f at X. Let $H \in \mathcal{O}$ such that $X + H \in \mathcal{O}$. Recall that $\phi \colon X \mapsto X^{-1}$ is differentiable and that its differential is

$$\mathrm{d}\phi(X)(H) = -X^{-1} H X^{-1}.$$

   Now, using the chain rule,

$$\begin{aligned} \mathrm{d}f(X)(H) &= \text{Tr}(-X^{-1} H X^{-1}) + \text{Tr}(AH) \\ &= \langle -(X^{-1})^2 + A, H \rangle. \end{aligned}$$

It follows that $\mathrm{d}f(X^*) = 0$ is equivalent to $-(X^{*-1})^2 + A = 0$, i.e.,

$$\boxed{X^* = A^{-1/2}}$$

Remark : Since A is positive symetric, $A^{-1/2}$ is uniquely defined.
Finally, the optimal value of f is

$$\begin{aligned} F(X^*) &= \mathrm{Tr}(A^{1/2}) + \mathrm{Tr}(A^{1/2}), \\ &= 2\mathrm{Tr}(A^{1/2}). \end{aligned}$$

Exercise 8 (Penalty method).
Let $F\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a lower semi-continuous function, coercive on $\mathbb{R}^n$. Let C be a closed set of $\mathbb{R}^n$ with $\mathrm{dom}(f) \cap C \neq \varnothing$. We seek to solve the constrained problem

$$\begin{aligned} &\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad F(x) \\ &\text{s.t.} \qquad\quad x \in C. \end{aligned} \tag{$\mathcal{P}$}$$

Let $R\colon \mathbb{R}^n \longrightarrow \mathbb{R}^+$ be a lower semi-continuous function such that

$$R(x) = 0 \quad \Longleftrightarrow \quad x \in C.$$

R is called penalty function as it assigns a positive cost to any point that is not in the constraint set C. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a nondecreasing sequence of positive reals satisfying $\lim_{k\to+\infty} \gamma_k = +\infty$. We denote by $(\mathcal{P}_k)$ the following penalized problem :

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad F_{\gamma_k}(x) := F(x) + \gamma_k R(x). \tag{$\mathcal{P}_k$}$$

Show that :

1. For all $k \in \mathbb{N}$, $(\mathcal{P}_k)$ has at least one solution x_k.
2. The sequence $(x_k)_{n\in\mathbb{N}}$ is bounded.
3. Any cluster point of $(x_k)_{k\in\mathbb{N}}$ is a solution to $(\mathcal{P})$.
4. What can we say if F is strictly convex ?

Correction.

1. Let $n \in \mathbb{N}$. Since R is positive and $\gamma_k > 0$, we have
$$\forall x \in \mathbb{R}^n, \ F_{\gamma_k}(x) \geqslant F(x).$$
Thus F_{γ_k} is coercive, l.s.c. and proper on the closed set C : there exists at least one solution to $(\mathcal{P}_k)$.
2. Let $\overline{x} \in C$. Then for all $n \geqslant 0$, $F_{\gamma_k}(\overline{x}) = F(\overline{x})$. Moreover, by definition of x_k,
$$x_k \in \mathrm{lev}_{\leqslant F_{\gamma_k}(\overline{x})} F_{\gamma_k} = \mathrm{lev}_{\leqslant F(\overline{x})} F_{\gamma_k} \subset \mathrm{lev}_{\leqslant F(\overline{x})} F,$$
last inclusion being a consequence of
$$\forall x \in \mathbb{R}^n, \ F_{\gamma_k}(x) \geqslant F(x).$$
Since F is coercive, $\mathrm{lev}_{\leqslant F(\overline{x})} F$ is bounded. Therefore $(x_k)_{k\in\mathbb{N}}$ is bounded.
3. Because R^n is of finite dimension, we can extract a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ converging to $x^* \in \mathbb{R}^n$. We must show that $\overline{x} \in C$, $F(x^*) \leqslant F(\overline{x})$ and $x^* \in C$.

▷ <u>Show $\forall \overline{x} \in C,\ F(x^*) \leqslant F(\overline{x})$.</u>
We have previously shown that for all $\overline{x} \in C$, for all $k \in \mathbb{N}$,

$$F(x_{k_j}) \leqslant F(\overline{x}).$$

Since F is l.s.c.,

$$F(x^*) \leqslant \underline{\lim}\ F(x_{k_j}) \leqslant F(\overline{x}).$$

▷ <u>Show $x^* \in C$.</u>
We have for all $j \in \mathbb{N}$,

$$\begin{aligned} F_{\gamma_{k_j}}(x_{k_j}) &= F(x_{k_j}) + \gamma_{k_j} R(x_{k_j}) \\ &= F_{\gamma_0}(x_{k_j}) + (\gamma_{k_j} - \gamma_0) R(x_{k_j}) \\ &\geqslant \inf F_{\gamma_0} + (\gamma_{k_j} - \gamma_0) R(x_{k_j}), \end{aligned}$$

thus

$$\begin{aligned} \forall \overline{x} \in C,\ 0 \leqslant R(x_{k_j}) &\leqslant \frac{F_{\gamma_{k_j}}(x_{k_j}) - \inf F_{\gamma_0}}{\gamma_{k_j} - \gamma_0} \\ &\leqslant \frac{F(\overline{x}) - \inf F_{\gamma_0}}{\gamma_{k_j} - \gamma_0}. \end{aligned}$$

Passing to the limit infinimum when $k \to +\infty$, the right hand side term goes to 0 and the left hand side one to $R(x^*)$ (because R is l.s.c.). It follows that

$$R(x^*) = 0.$$

This proves that $x^* \in C$.

Finally x^* is a solution to $(\mathcal{P})$.

4. If F is strictly convex, there is an unique solution to problem $(\mathcal{P})$. Hence, $(x_k)_{k \in \mathbb{N}}$ is a bounded sequence with a single cluster point. We can conclude that $(x_k)_{k \in \mathbb{N}}$ converges.

Optimization – Exercises

Day 2

Exercise 1 (Convergence fixed step gradient descent algorithm).
For all $x \in \mathbb{R}^n$ we define the function f by

$$f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle,$$

where $A \in \mathcal{S}_n^{++}(\mathbb{R})$, with eigenvalues $(\lambda_i)_{1\leqslant i\leqslant n}$ verifying

$$0 < \lambda_1 \leqslant \ldots \leqslant \lambda_n,$$

and $b \in \mathbb{R}^n$. It has already been seen in exercise 4 that f admits a unique minimizer x^*, which is the solution to the linear system $Ax = b$.
The fixed step gradient descent algorithm is given by

$$\begin{cases} x_0 \in \mathbb{R}^n, \\ x_{k+1} = x_k - \gamma \nabla f(x_k). \end{cases}$$

Show the algorithm converges to x^* for any step $\gamma \in \left]0, \frac{2}{\lambda_n}\right[$. Give the step γ that ensures the fastest convergence.

Correction. Recall $\|\cdot\|$ denotes the euclidian norm of $\mathbb{R}^n$. Let $k \in \mathbb{N}^*$. By definition of $(x_k)_{k\in\mathbb{N}}$

$$\begin{aligned}\|x_{k+1} - x^*\| &= \|x_k - \gamma\nabla f(x_k) - x^*\| \\ &= \|(x_k - x^*) - (\gamma\nabla f(x_k) - \gamma\nabla f(x^*))\|,\end{aligned}$$

since x^* verifies $\nabla f(x^*) = 0$. It follows that

$$\begin{aligned}\|x_{k+1} - x^*\| &= \|(x_k - x^*) - \gamma(Ax_k - Ax^*)\| \\ &= \|(\mathrm{I}_n - \gamma A)(x_k - x^*)\| \\ &\leqslant \|\mathrm{I}_n - \gamma A\|\|x_k - x^*\|.\end{aligned}$$

Since $\mathrm{I}_n - \gamma A$ is symmetric, $\|\mathrm{I}_n - \gamma A\| = \rho(\mathrm{I}_n - \gamma A)$, where $\rho(X) = \sup\{|\lambda|,\ \lambda \text{ eigenvalue of } X\}$. Hence

$$\|\mathrm{I}_n - \gamma A\| = \max_{1\leqslant j\leqslant n}\{|1 - \gamma\lambda_j|\}$$

Now if $\gamma \in \left]0, \frac{2}{\lambda_n}\right[$, we easily show that for all $i \in \{1, \ldots, n\}$

$$1 > 1 - \gamma\lambda_i > -1,$$

i.e.

$$|1 - \gamma\lambda_i| < 1.$$

Then

$$\|\mathrm{I}_n - \gamma A\| < 1.$$

By recurrence

$$\|x_k - x^*\| \leqslant \|\mathrm{I}_n - \gamma A\|^k \|x_0 - x^*\| \underset{k \to +\infty}{\longrightarrow} 0,$$

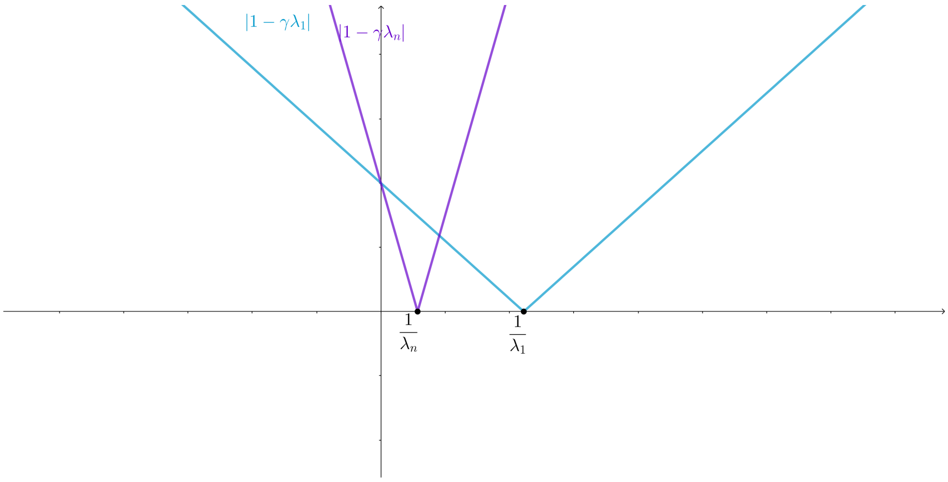
which proves the algorithm converges.
Best step γ. To find the best step $\gamma \in \left]0, \frac{2}{\lambda_n}\right[$, we first note that

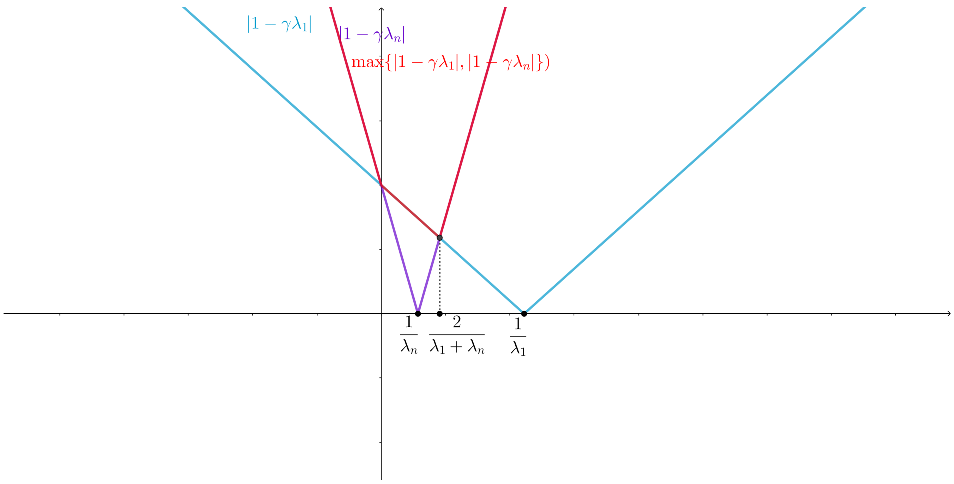
$$\|\mathrm{I}_n - \gamma A\| = \max\{|1 - \gamma\lambda_1|, |1 - \gamma\lambda_n|\}.$$

Then, drawing the function $\gamma \mapsto \max\{|1 - \gamma\lambda_1|, |1 - \gamma\lambda_n|\}$, we observe the minimum is reached at $\gamma = \frac{1}{\lambda_1 + \lambda_n}$. Indeed, the intersection point γ_{opt} of the line $\gamma \mapsto 1 - \gamma\lambda_1$ with the line $\gamma \mapsto \gamma\lambda_n - 1$ can be found solving

$$1 - \gamma\lambda_1 = \gamma\lambda_n - 1.$$

This value of γ ensures the fastest convergence.





Exercise 2 (Convergence of Uzawa method).
Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable α-strongly convex function and let $C \in \mathbb{R}^{m\times n}$, $d \in \mathbb{R}^m$. We

propose to study the convergence of Uzawa method towards a solution to the following problem :

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & Cx \leqslant d, \end{aligned} \tag{$\mathcal{P}$}$$

where the set $\{x \in \mathbb{R}^N \mid Cx \leqslant d\}$ is assumed to be nonempty. Let $\rho > 0$. Uzawa algorithm generates sequences $(x_k)_{k\in\mathbb{N}} \in (\mathbb{R}^n)^{\mathbb{N}}$ and $(\lambda_k)_{k\in\mathbb{N}} \in (\mathbb{R}^m)^{\mathbb{N}}$ according to the following iterations :

$$\boxed{\begin{cases} x_k = \underset{x\in\mathbb{R}^n}{\text{argmin}}\ f(x) + \langle \lambda_k, Cx - d\rangle, \\ \lambda_{k+1} = \max\left(\lambda_k + \rho(Cx_k + d), 0\right). \end{cases}}$$

1. Explain why Problem ($\mathcal{P}$) admits a unique solution and why the algorithm is well defined.
2. (i) Write the Lagrangian $\mathcal{L} \colon \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ for Problem ($\mathcal{P}$).
   (ii) Show that for any $x \in \mathbb{R}^n$,

$$\left(\lambda^* = \underset{\lambda\in[0,+\infty)^m}{\text{argmax}}\ \mathcal{L}(x,\lambda)\right) \iff \left((\forall \rho > 0) \quad \lambda^* = \mathrm{p}_+(\lambda^* + \rho(Cx - d))\right),$$

   where p_+ denotes the projection on $[0, +\infty)^m$.
   (iii) Let (x^*, λ^*) be a saddle point of $\mathcal{L}$. Show that the following holds :

$$\begin{cases} \nabla f(x_k) - \nabla f(x^*) + C^\top(\lambda_k - \lambda) = 0 \\ \|\lambda_{k+1} - \lambda^*\| \leqslant \|\lambda_k - \lambda^* + \rho C(x_k - x^*)\|. \end{cases} \tag{$\star$}$$

3. Using ($\star$), show the convergence of the sequence $(x_k)_{k\in\mathbb{N}}$ to x^* when ρ satisfies

$$0 < \rho < \frac{2\alpha}{\|C\|^2}. \tag{$\star\star$}$$

Correction.

1. f is strongly convex, continuous, and the set $\{x \in \mathbb{R}^N \mid Cx \leqslant d\}$ is nonempty, closed. Therefore, Problem ($\mathcal{P}$) admits a unique solution.
   If you don't already know this result, it can easily be proven the following way, using the differentiability of f.
   — **Existence**. The strong convexity of f implies that for all $(x, y) \in (\mathbb{R}^n)^2$ (see Exercise 3 in class notes),

$$\begin{aligned} f(x) &\geqslant f(y) + \langle \nabla f(y), x - y\rangle + \frac{\alpha}{2}\|x - y\|^2, \\ &\geqslant f(y) - \|\nabla f(y)\|\|x - y\| + \frac{\alpha}{2}\|x - y\|^2, \end{aligned}$$

   where we used Cauchy-Schwarz inequality. The minoring term is a polynomial function of degree 2 with a positive dominant coefficient. We deduce that f is coercive. Since the set $\{x \in \mathbb{R}^N \mid Cx \leqslant d\}$ is nonempty, and closed, the existence of a global minimizer follows from the course.
   — **Unicity**. A strongly convex function is strictly convex.
2. (i) For all $(x, \lambda) \in \mathbb{R}^n \times [0, +\infty)^m$,

$$\mathcal{L}(x, \lambda) = f(x) + \lambda^\top(Cx - d).$$

(ii) Let $x \in \mathbb{R}^n$,

$$\left(\lambda^* = \underset{\lambda\in[0,+\infty)^m}{\text{argmax}} \ \mathcal{L}(x,\lambda)\right) \underset{\text{Euler}}{\Longleftrightarrow} (\forall \lambda \in [0,+\infty)^m \quad \langle \nabla_2 \mathcal{L}(x,\lambda^*), \lambda - \lambda^* \rangle \leqslant 0)$$
$$\Longleftrightarrow (\forall \lambda \in [0,+\infty)^m, \forall \rho > 0 \quad \langle \rho(Cx-d), \lambda - \lambda^* \rangle \leqslant 0)$$
$$\Longleftrightarrow (\forall \lambda \in [0,+\infty)^m, \forall \rho > 0 \quad \langle \lambda^* + \rho(Cx-d) - \lambda^*, \lambda - \lambda^* \rangle \leqslant 0)$$
$$\underset{\text{proj.}}{\Longleftrightarrow} (\forall \rho > 0 \quad \lambda^* = \mathrm{p}_+(\lambda^* + \rho(Cx-d)))\,.$$

(iii) Since (x^*,λ^*) is a saddle point of $\mathcal{L}$, the following holds :

$$x^* = \inf_{x\in\mathbb{R}^n} \mathcal{L}(x,\lambda^*) \quad \text{and} \quad \lambda^* = \sup_{\lambda\geqslant 0} \mathcal{L}(x^*,\lambda).$$

We deduce from the previous question that

$$\begin{cases} \nabla f(x^*) + C^\top \lambda^* = 0, \\ \lambda^* = \mathrm{p}_+(\lambda^* + \rho(Cx^* - d)). \end{cases}$$

It is clear from the definition of the algorithm that we have similar relations for the iterates :

$$\begin{cases} \nabla f(x_k) + C^\top \lambda_k = 0, \\ \lambda_{k+1} = \mathrm{p}_+(\lambda_k + \rho(Cx_k - d)). \end{cases}$$

Finally, combining these inequalities and recalling that the projection operator p_+ is 1-Lipschitz, we obtain ($\star$).

3. For all $k \in \mathbb{N}$,

$$\begin{aligned} \|\lambda_{k+1} - \lambda^*\|^2 &\leqslant \|\lambda_k - \lambda^* + \rho C(x_k - x^*)\|^2 \\ &= \|\lambda_k - \lambda^*\|^2 + \rho^2\|C(x_k - x^*)\|^2 + 2\rho\langle C^\top(\lambda_k - \lambda^*), x_k - x^*\rangle \\ &= \|\lambda_k - \lambda^*\|^2 + \rho^2\|C(x_k - x^*)\|^2 + 2\rho\langle \nabla f(x^*) - \nabla f(x_k), x_k - x^*\rangle. \end{aligned}$$

Now, since f is α-strongly convex,

$$\forall (x,y) \in (\mathbb{R}^n)^2 \quad \langle \nabla f(x) - \nabla f(y), x - y\rangle \geqslant \alpha\|x-y\|^2.$$

Therefore,

$$\begin{aligned} \|\lambda_{k+1} - \lambda^*\|^2 &\leqslant \|\lambda_k - \lambda^*\|^2 + \rho^2\|C\|^2\|x_k - x^*\|^2 - \alpha 2\rho\|x_k - x^*\|^2, \\ &= \|\lambda_k - \lambda^*\|^2 + \rho(\rho\|C\|^2 - 2\alpha)\|x_k - x^*\|^2. \end{aligned}$$

Therefore, when the condition ($\star\star$) is met, the sequence $(\|\lambda_k - \lambda^*\|)_{k\in\mathbb{N}}$ is decreasing and bounded from below, thus it converges. It follows that $\|\lambda_k - \lambda^*\|^2 - \|\lambda_{k+1} - \lambda^*\|^2 \underset{k\to+\infty}{\longrightarrow} 0$ and

$$\rho(2\alpha - \rho\|C\|^2)\|x_k - x^*\| \leqslant \|\lambda_k - \lambda^*\|^2 - \|\lambda_{k+1} - \lambda^*\|^2 \underset{k\to+\infty}{\longrightarrow} 0.$$

Additional remarks.

— If problem $(\mathcal{P})$ is feasible and (x^*,λ^*) is a saddle point of $\mathcal{L}$, then x^* is a solution to the primal problem $(\mathcal{P})$.

— Let x^* be a solution to the primal problem $(\mathcal{P})$. Then, for a convex problem (convex objective function + affine constraint functions) for which the constraints are qualified (Slater), there exists a $\lambda^* \in [0,+\infty)^m$ such that (x^*,λ^*) is a saddle point of $\mathcal{L}$. See class notes, page 11.

Exercise 3 (Optimization with equality constraints).
Find the points (x, y, z) de $\mathbb{R}^3$ which belong to H_1 and H_2 and which are the closest to the origin.

$$(H_1) \ : \ 3x + y + z = 5,$$
$$(H_2) \ : \ x + y + z = 1.$$

1. Write the problem as an optimization problem.
2. What can you say about existence of solutions ? Unicity ?
3. Solve the optimization problem using the Slater conditions.

Correction.

1. We can write the problem as

$$\begin{array}{ll} \underset{(x,y,z)\in\mathbb{R}^3}{\text{minimize}} & f(x,y,z) \\ \text{subject to} & g_1(x,y,z) = 0 \\ & g_2(x,y,z) = 0, \end{array}$$

with $f(x,y,z) := x^2+y^2+z^2$, $g_1(x,y,z) := 3x+y+z-5$ and $g_2(x,y,z) := x+y+z-1$.

2. Let
$$C = \{(x,y,z) \in \mathbb{R}^3 \mid g_1(x,y,z) = 0 \text{ and } g_2(x,y,z) = 0\}.$$
The problem is feasible because $\operatorname{dom}(f) \cap C \neq \varnothing$ (notice $(2, 0, -1) \in C$). Moreover, fonction f is coercive, lower semi-continuous on the closed set C, hence there exists at least one global minimizer. Finally, f is strictly convex on the convex set C, the minimizer is therefore unique.

3. — f is convex, continuously differentiable,
   — g_1 et g_2 are affine,
   — Slater condition holds : consider the point $(\overline{x}, \overline{y}, \overline{z}) = (2, 0, -1)$.

   We deduce from KKT theorem in the convexe case (see page 11 in the course) that (x, y, z) is a minimizer of f over C if, and only if, there exists $(\mu_1, \mu_2) \in \mathbb{R}^2$ such that

$$\nabla f(x,y,z) + \mu_1 \nabla g_1(x,y,z) + \mu_2 \nabla g_2(x,y,z) = 0, \qquad \text{(Stationarity condition)}$$

   and

$$g_1(x,y,z) = 0 \quad \text{et} \quad g_2(x,y,z) = 0. \qquad \text{(Constraints)}$$

   These two conditions are equivalent to the following system of equations

$$\begin{cases} 2x + 3\mu_1 + \mu_2 & = 0 \\ 2y + \mu_1 + \mu_2 & = 0 \\ 2z + \mu_1 + \mu_2 & = 0 \\ 3x + y + z & = 5 \\ x + y + z & = -1. \end{cases}$$

   The unique solution of this system is

$$\begin{cases} x & = 0 \\ y & = -1/2 \\ z & = -1/2 \\ \mu_1 & = -7/2 \\ \mu_2 & = 9/2. \end{cases}$$

   <u>Conclusion</u> : the unique minimizer of f over C is

$$\boxed{(\hat{x}, \hat{y}, \hat{z}) = (2, -1/2, -1/2).}$$

Exercise 4 (Optimization with inequality constraints).
Solve the following optimization problem :

$$\begin{aligned} \underset{(x,y)\in\mathbb{R}^2}{\text{minimize}} \quad & x^4 + 3y^4 \\ \text{subject to} \quad & x^2 + y^2 \geqslant 1. \end{aligned}$$

Correction. Let $f(x,y) := x^4 + 3y^4$, $h(x,y) := -x^2 - y^2 + 1$ and

$$C = \{x \in \mathbb{R}^2 \;\mid\; h(x,y) \leqslant 0\}.$$

1. The problem is feasible since $C \cap \operatorname{dom}(f) \neq \varnothing$. Function f is coercive since

$$\begin{aligned} f(x,y) &\geqslant (2x^2 - 1) + 3(y^2 - 1) \\ &\geqslant 2\|(x,y)\|^2 - 4 \underset{\|(x,y)\|\to+\infty}{\longrightarrow} +\infty. \end{aligned}$$

   Moreover f is *l.s.c* on the closed set C : this ensures the existence of global minimizer to f on C.

   ⚠ We cannot say anything about the unicity of the solution even though f is strictly convex on $\mathbb{R}^n$, because C is not convex.

2. To find the solution(s) to the problem, we are going to apply KKT theorem. It will give us a necessary optimality condition.
   - f and h are continuously differentiable.
   - The Mangasarian Fromovitz qualification of constraints holds for all $(x,y) \in C$ because $\nabla h(x,y) = (2x, 2y) \neq 0$ for all $(x,y) \in C$.

   The hypothesis of KKT theorem are verified. Let x be a local minimizer of f on C : there exists $\lambda \in \mathbb{R}^+$ such that

$$\nabla f(x) + \lambda \nabla h(x) = 0, \qquad \text{(Stationarity condition)}$$

   and

$$\lambda h(x) = 0, \qquad \text{(Complementary slackness condition)}$$

   and

$$h(x) \leqslant 0. \qquad \text{(Constraints)}$$

   These three conditions are equivalent to

$$\begin{cases} 4x^3 - 2\lambda x & = 0 \\ 12y^3 - 2\lambda y & = 0 \\ \lambda(x^2 + y^2 - 1) & = 0 \\ x^2 + y^2 & \geqslant 1. \end{cases}$$

   If $\lambda = 0$, there is no solution to this system. Hence $\lambda > 0$ and the system is equivalent to

$$\begin{cases} 4x^3 - 2\lambda x & = 0 \\ 12y^3 - 2\lambda y & = 0 \\ x^2 + y^2 - 1 & = 0 \end{cases} \qquad (\star)$$

   The first two equations of $(\star)$ give the following couples $(x,y) \in \mathbb{R}^2$

$$\begin{aligned} x &\in \left\{\pm\sqrt{\frac{\lambda}{2}}, 0\right\} \\ y &\in \left\{\pm\sqrt{\frac{\lambda}{6}}, 0\right\}. \end{aligned}$$

Considering now the third equation, we deduce the solutions to the system $(\star)$ are the couples $(x, y) \in \mathbb{R}^2$

$$\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right) \tag{1}$$
$$(\pm 1, 0) \tag{2}$$
$$(0, \pm 1)\,. \tag{3}$$

We now need to select among these couples those who are global minimizers, that is to say those who give the smallest value of f.

- ▷ Among couples (x, y) of the form (1), $f(x, y) = \frac{9}{16} + 3\frac{1}{16} = \frac{3}{4}$.
- ▷ Among couples (x, y) of the form (2), $f(x, y) = 1$.
- ▷ Among couples (x, y) of the form (3), $f(x, y) = 3$.

Conclusion : the solutions to the optimization problem are the couples

$$\boxed{(x, y) = \left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right).}$$

Exercise 5 (Optimization with equality and inequality constraints).
Let $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ be defined by

$$f(p_1, \ldots, p_k) = \sum_{i=1}^{k} p_i^2.$$

Maximize f on the simplex Λ_k of $\mathbb{R}^k$

$$\Lambda_k := \left\{ p = (p_1, \ldots, p_k) \in \mathbb{R}^k \mid p_i \geqslant 0 \text{ for all } i, \text{ and } \sum_{i=1}^{k} p_i = 1 \right\}.$$

Correction.

1. We start by showing the existence of solutions to the problem. The function f is continuous on Λ_k and Λ_k is compact : indeed,

$$\Lambda_k = \left(\bigcap_{i=1}^{k} h_i^{-1}(]-\infty, 0]) \right) \bigcap g^{-1}(\{0\}),$$

where for any $i \in \{1, \ldots, k\}$, for any $p \in \mathbb{R}^k$,

$$h_i(p) = -p_i,$$
$$g(p) = \sum_{i=1}^{k} p_i - 1.$$

So Λ_k is closed. Moreover Λ_k is bounded because it is included in $\left\{ p = (p_1, \ldots, p_k) \in \mathbb{R}^k \mid 0 \leqslant p_i \leqslant 1 \right\}$.
We deduce that f reaches its maximum on Λ_k.

2. Let us find the solutions to this problem.
   - ▷ f, $(h_i)_{1 \leqslant i \leqslant k}$ and g are continuously differentiable,

▷ **Mangasarian-Fromovitz's constraint qualification**. We check that the constraints are qualified at all $p \in \Lambda_k$. Let $p \in \Lambda_k$. Denote

$$J(p) = \{i \in \{1, \dots, k\} \mid h_i(p) = 0\} = \{i \in \{1, \dots, k\} \mid p_i = 0\}$$

Since $(0, \dots, 0) \notin \Lambda_k$, necessarily, there exists $\ell \in \{1, \dots, k\}$ such that $p_\ell \neq 0$. Set $z \in \mathbb{R}^k$ with $z_i = 1$ if $i \neq \ell$, $z_\ell = -(k-1)$. Then

$$\begin{aligned} \langle \nabla g(p), z \rangle = 0 \\ \forall j \in J(p),\ \langle \nabla h_j(p), z \rangle < 0. \end{aligned}$$

The constraints are therefore qualified at p.

Let $p \in \mathbb{R}^k$ be a local minimum of $-f$ on Λ_k. According to the KKT theorem, there exist $\lambda = (\lambda_1, \dots, \lambda_k) \in (\mathbb{R}^+)^k$ and $\mu \in \mathbb{R}$ such that

$$-\nabla f(p) + \mu \nabla g(p) + \sum_{j=1}^{k} \lambda_j \nabla h_j(p) = 0, \qquad \text{(Stationarity condition)}$$

and

$$\forall j \in \{1, \dots, q\},\ \lambda_j h_j(p) = 0, \qquad \text{(Complementary slackness)}$$

and

$$\begin{cases} & g(p) = 0 \\ \forall j \in \{1, \dots, k\}, & h_j(p) \leqslant 0. \end{cases} \qquad \text{(Constraints)}$$

These three conditions boil down to the following system

$$\begin{cases} \forall i \in \{1, \dots, k\},\ -2p_i - \lambda_i + \mu = 0, \\ \forall i \in \{1, \dots, k\},\ \lambda_i p_i = 0, \\ \forall i \in \{1, \dots, k\},\ p_i \geqslant 0, \\ \sum_{i=1}^k p_i = 1. \end{cases}$$

Let $J = \{i \in \{1, \dots, k\} \mid p_i = 0\}$. Notice again that $|J| \neq k$ because $p = (0, \dots, 0)$ is not in Λ_k. The system of equations implies that

$$\sum_{i=1}^{n} (\lambda_i - \mu) = -2$$

i.e.

$$\sum_{i \in J} \lambda_i = -2 + k\mu \tag{4}$$

Moreover, for all $i \in J$, $p_i = 0$ so from the first equation of the system, $i = \mu$. Thus (4) is rewritten

$$|J|\mu = -2 + k\mu$$

i.e.

$$\boxed{\mu = \frac{2}{k - |J|}}.$$

We deduce

$$p_i = \begin{cases} 0 & \text{if } i \in J \\ \frac{1}{k-|J|} & \text{otherwise}. \end{cases} \tag{5}$$

Thus the global maximizers of f on Λ_k are to be sought among the p of the form (5), with $|J| \in \{0, \ldots, k-1\}$. Let us select the p of this form maximizing f. We have

$$\begin{aligned} f(p) &= \sum_{i \notin J} p_i^2 \\ &= |J^c| \times \frac{1}{(k-|J|)^2} \\ &= (k-|J|) \times \frac{1}{(k-|J|)^2} \\ &= \frac{1}{k-|J|}. \end{aligned}$$

Thus f is maximal if $|J| = k-1$. Finally, the solutions of the optimization problem are $(e_i)_{(1 \leqslant i \leqslant k)}$, where e_i denotes the i-th vector of the canonical basis of $\mathbb{R}^k$. In other words, the solutions are vertices of the simplex.

Exercise 6 (Characterization of $\mathrm{SO}_n(\mathbb{R})$).
We denote $\mathrm{SO}_n(\mathbb{R}) = \{M \in \mathbb{R}^{n \times n} \mid M \text{ is orthogonal and } \det(M) = 1\}$ and $\mathrm{SL}_n(\mathbb{R}) = \{M \in \mathbb{R}^{n \times n} \mid \det(M) = 1\}$. Show $\mathrm{SO}_n(\mathbb{R})$ is exactly composed of the matrices of $\mathrm{SL}_n(\mathbb{R})$ which minimize the Euclidean norm of $\mathbb{R}^{n \times n}$, i.e.

$$\forall M \in \mathbb{R}^{n \times n}, \ \|M\| = \sqrt{\mathrm{Tr}(M^\top M)}.$$

Correction 1. We must show that

$$\mathrm{SO}_n(\mathbb{R}) = \left\{ M \in \mathbb{R}^{n \times n} \ \middle| \ \|M\|^2 = \inf_{A \in \mathrm{SL}_n(\mathbb{R})} \|A\|^2 \right\}.$$

- ▷ Existence of a minimizer. Let $g \colon M \mapsto \det(M) - 1$. Since g is continuous, $\mathrm{SL}_n(\mathbb{R}) = g^{-1}(\{0\})$ is closed in $\mathbb{R}^{n \times n}$. In addition, $f \colon M \mapsto \|M\|^2$ is continuous and coercive. Thus f admits a minimizer on $\mathrm{SL}_n(\mathbb{R})$.
- ▷ Characterize the minimizers. Let M be a minimizer of f on $\mathrm{SL}_n(\mathbb{R})$. Then form the Lagrange multiplier theory, there exists $\mu \in \mathbb{R}$ such that

$$\nabla f(M) = \mu \nabla g(M).$$

Using the differential of the determinant function, it follows that :

$$2M = \mu \mathrm{Com}(M), \tag{6}$$

where $\mathrm{Com}(M)$ denotes the comatrix. Applying det on both sides and using $\det(M) = 1$ yields

$$\begin{aligned} 2^n &= \mu^n \det(\mathrm{Com}(M)) \\ &= \mu^n \det((M^{-1})^\top) \\ &= \mu^n \frac{1}{\det(M)} \\ &= \mu^n. \end{aligned}$$

Hence $\mu = 2$ or $\mu = -2$. Now, multiplying (6) by $M^\top$, we obtain :

$$\begin{aligned} 2MM^\top &= \mu \mathrm{Com}(M) M^\top \\ &= \mu \mathrm{I}_n. \end{aligned}$$

Taking the trace on both sides implies that $\mu > 0$. Finally $\mu = 2$, and $MM^\top = \mathrm{I}_n$. Thus $M \in \mathrm{SO}_n(\mathbb{R})$.

▷ Check. Conversely, let $M \in \mathrm{SO}_n(\mathbb{R})$. Then

$$\|M\|^2 = \mathrm{Tr}(M^\top M) = n.$$

Thus f in constant on $\mathrm{SO}_n(\mathbb{R})$. Since we know f has at least one minimizer in $\mathrm{SO}_n(\mathbb{R})$, we deduce that any matrix of $\mathrm{SO}_n(\mathbb{R})$ is a minimizer of f.